TOPOLOGY III - MID-SEMESTRAL EXAM.

Time: 3 hours Max. marks: 60

Answer all questions. You may use results proved in class after correctly quoting them. Any other claim must be accompanied by a proof.

(1) Let X be a space and let $S_*(X)$ denote its singular chain complex. For an abelian group G, define

$$h_n(X;G) = H_n(\mathrm{Hom}(G, S_*(X))$$

where $\mathrm{Hom}(G, S_*(X))$ denotes the chain complex whose n-th chain group is $\mathrm{Hom}(G, S_n(X))$ with he obvious boundary map. Compute the groups $h_n(X;G)$ when $G = \mathbb{Z}, \mathbb{Z}_2, \mathbb{Q}$. [6]

(2) Let $X = \mathbb{R}P^2 \times S^3$. Compute the groups $H^i(X;G)$ for $G = \mathbb{Z}, \mathbb{Z}_2$. [12]

(3) Define the notion of the degree of a map between two connected closed oriented manifolds. Show that if X is a connected closed orientable n-manifold, then there exists a map $f : X \longrightarrow S^n$ of degree 1. [2+10]

(4) Let X be a connected closed orientable n-manifold. Assume that there exists a map $f : S^n \longrightarrow X$ of degree $k \neq 0$. Prove that $H_i(X;\mathbb{Q}) \cong H_i(S^n;\mathbb{Q})$. [12]

(5) Let $(X, x_0), (Y, y_0)$ be based spaces with X, Y locally compact Hausdorff. Prove that there is bijection

$$[\Sigma(X, x_0), (Y, y_0)] \longrightarrow [(X, x_0), \Omega(Y, y_0)]$$

between the homotopy sets.

(6) Define the term : fibration. Show that the projection

$$\{(x, y) \in \mathbb{R}^2 \, : \, x \in [0, 1], y \leq 1 - x\} \longrightarrow [0, 1]$$

to the first factor is a fibration. [2+4]

(7) Define the term: fiber bundle. Show that there are fiber bundles

$$S^1 \hookrightarrow S^{2n+1} \longrightarrow \mathbb{C}P^n, \quad SO(n-1) \hookrightarrow SO(n) \longrightarrow S^{n-1}$$

Use the above to compute

(a) $\pi_i(\mathbb{C}P^n)$ (in terms of those of the homotopy groups of the spheres) and

(b) $\pi_i(SO(n))$ for $i = 1, 2$ $(n \geq 2)$. [2 + 4+6]